Introduction to covariance analysis technique – in contrast to coincidence analysis

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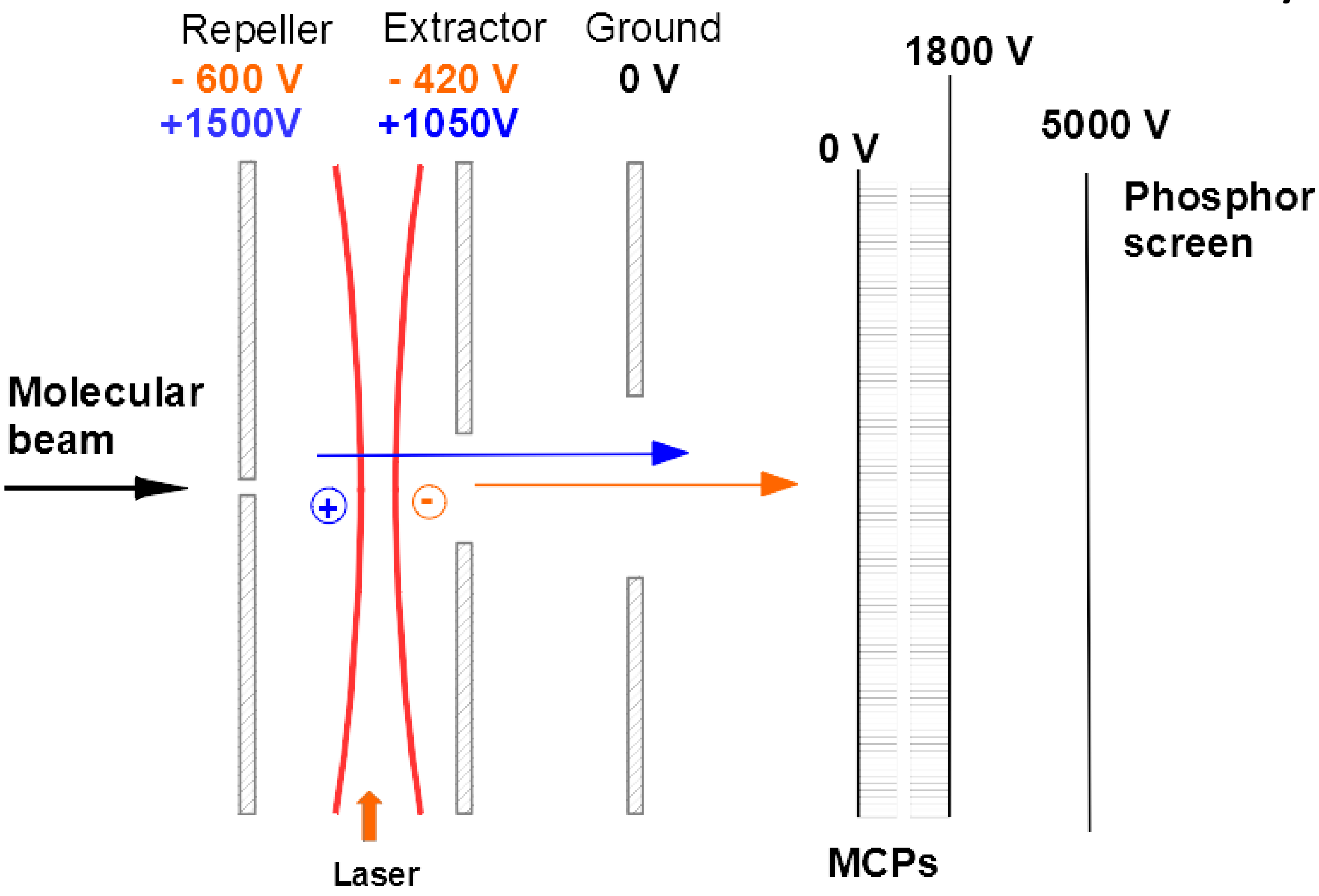
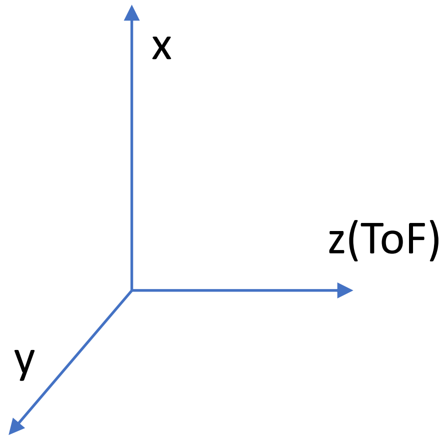
2021/09/14

1. Power of coincidence analysis





- Multiple particles coincidence
- multiple dimensional measurements

$$i^{+}: p_x, p_y, p_z\ \left(\frac{m}{q}\right)$$

$$e^{-}:\ p_x, p_y$$

1. Power of coincidence analysis

$$Data = N(particles)$$

ToF

$$= N(e) + N(i_1) + N(i_2) + \cdots$$

Differential measurements

$$= N(e, p_x, p_y) + N(i_1, p_x, p_y, p_z) + \cdots$$

Abel Inversion

$$= N(e, KE\ , \theta) + N(i_1, KE, \theta) + \cdots$$

Coincidence

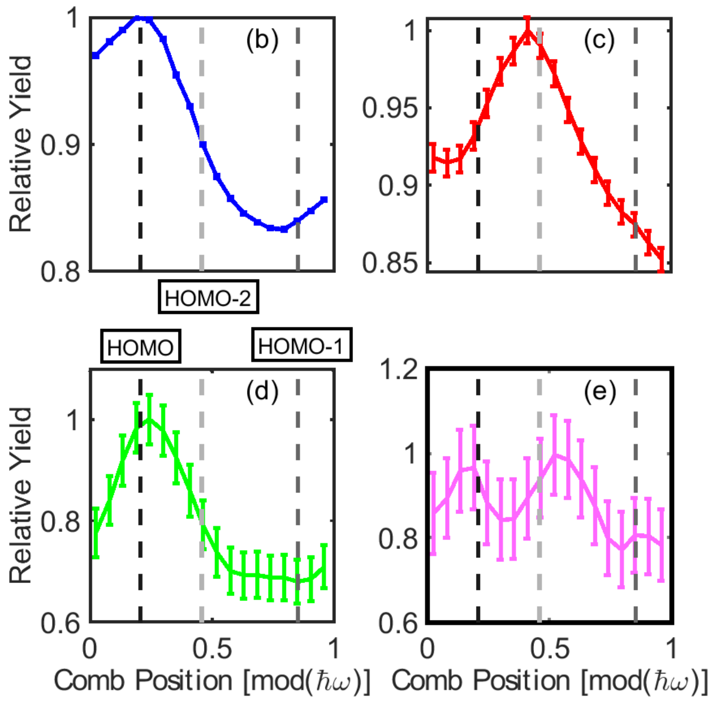
$$= N(e, p_x, p_y | i_1) + N(i_1, i_2) + \cdots$$

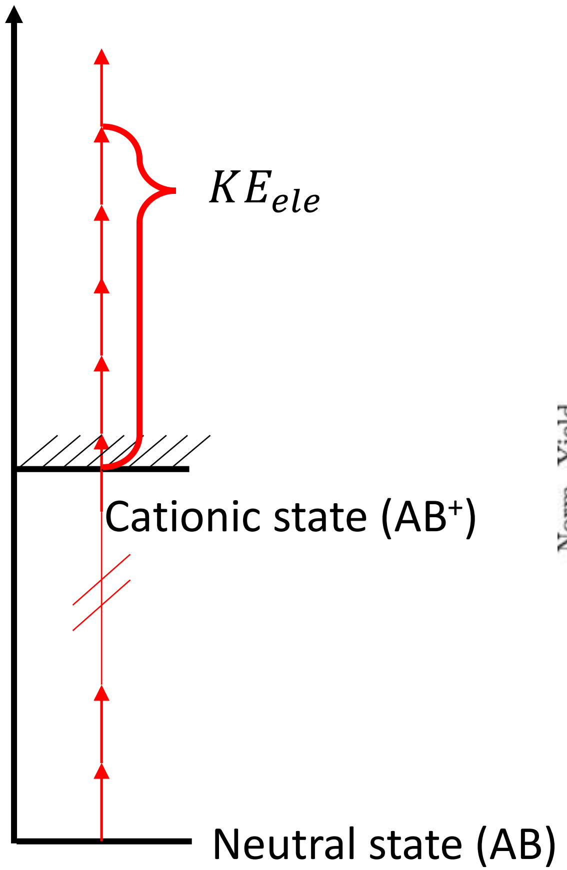
Electron's $p_x p_y$ image in coincidence with ion specie 1

Abel Inversion or other analysis

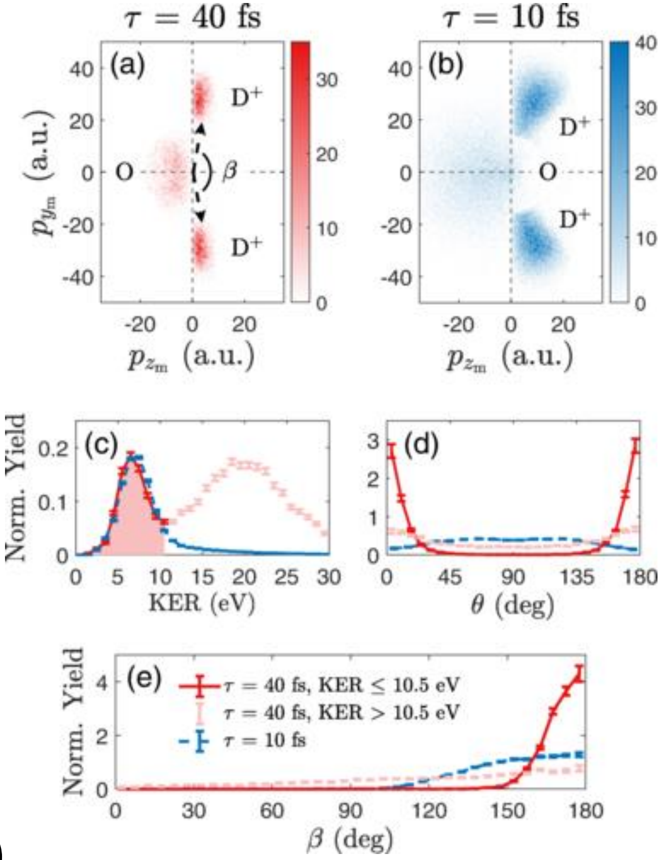
1. Power of coincidence analysis

Photoelectron in coincidence with different ions



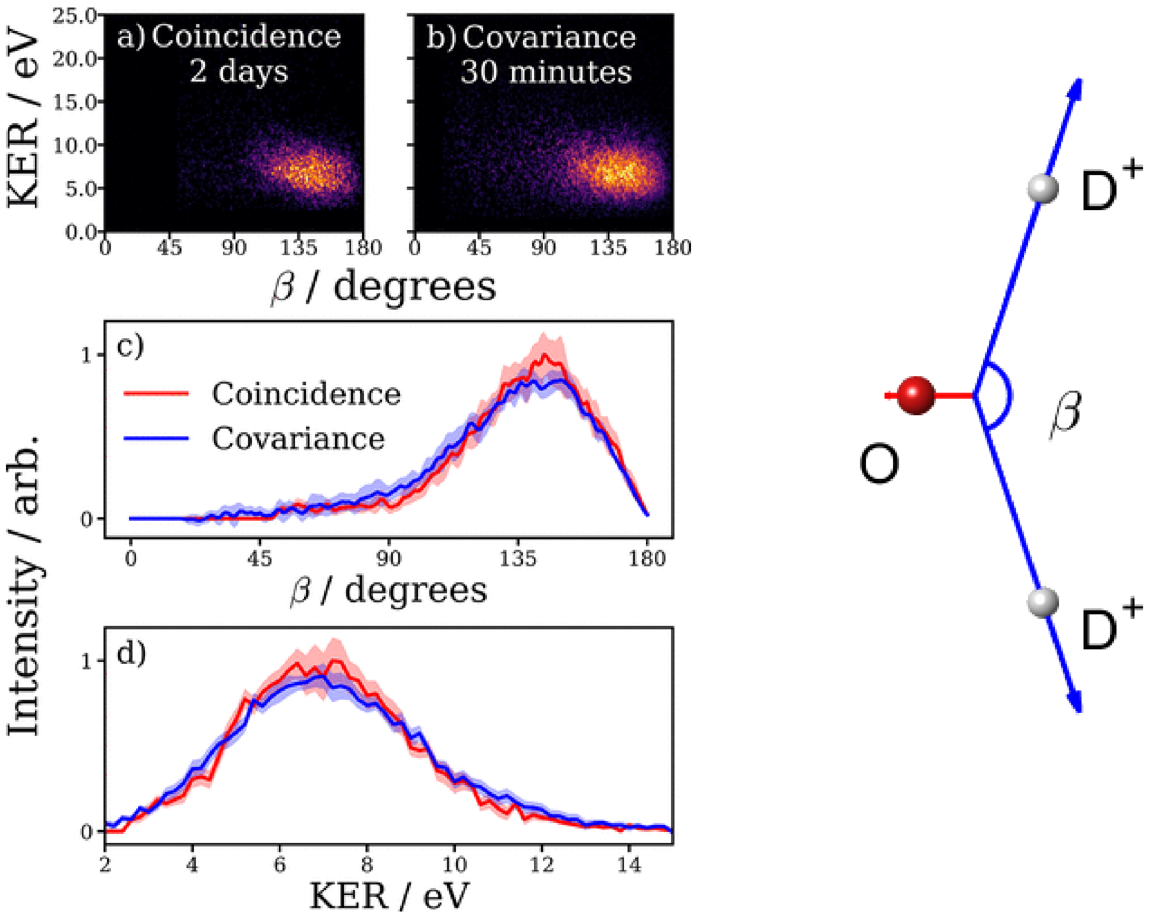


Photoion correlations arised from multi-ionization



Cheng, Chuan, et al. "Momentum-resolved above-threshold ionization of deuterated water." *Physical Review A* 102.5 (2020): 052813
Howard, A. J., et al. "Strong-field ionization of water: Nuclear dynamics revealed by varying the pulse duration." *Physical Review A* 103.4 (2021): 043120.

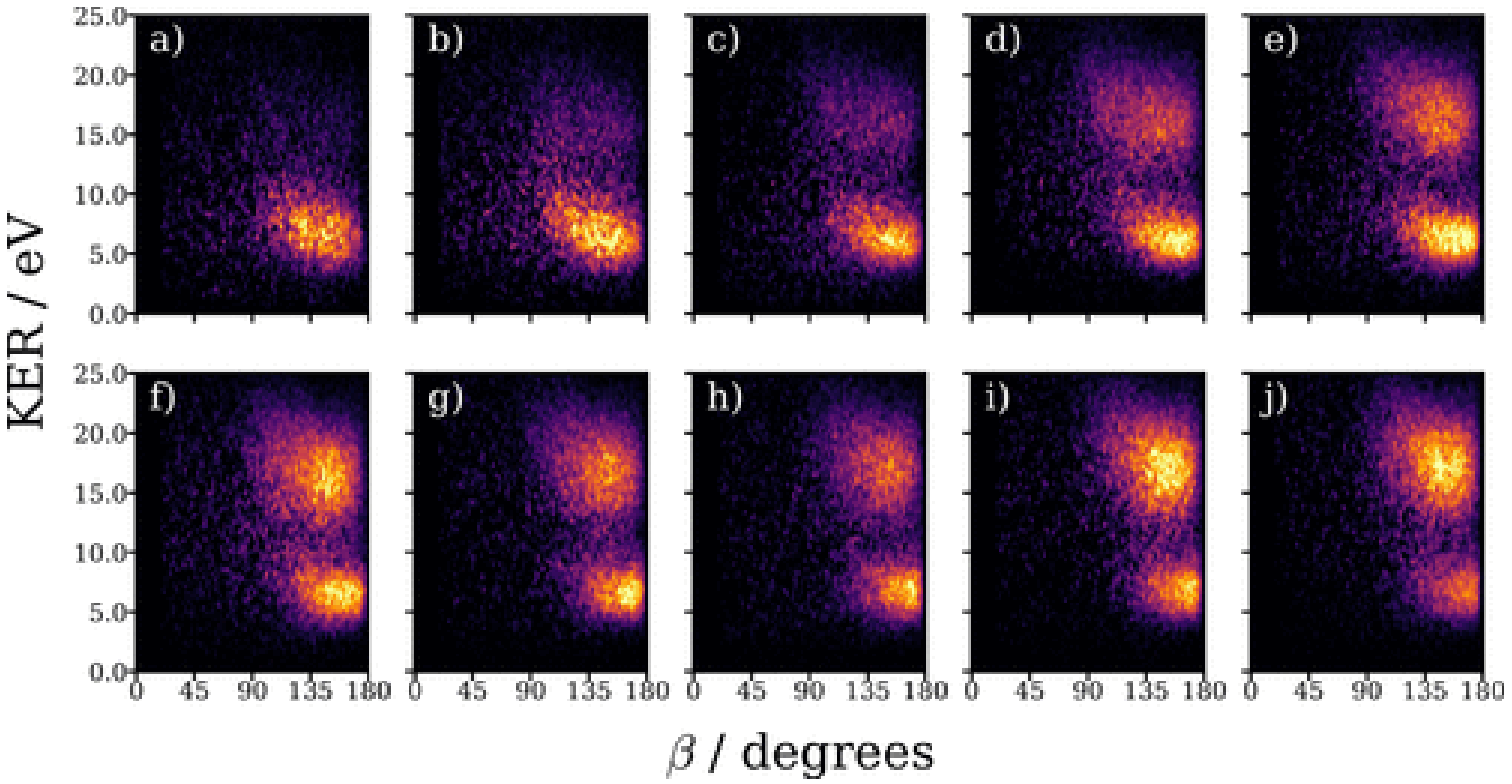
1. Power of coincidence analysis – coincidence power for covariance price



- Coin vs covar analysis in ion-ion events
- Boost factor = 2days/30mins ≈ 100

Allum, Felix, et al. "Multi-Particle Three-Dimensional Covariance Imaging:“Coincidence” Insights into the Many-Body Fragmentation of Strong-Field Ionized D2O." *The Journal of Physical Chemistry Letters* 12 (2021): 8302-8308.

1. Power of coincidence analysis – coincidence power for covariance price



- Coin vs covar analysis in ion-ion events
- Boost factor = 2days/30mins ≈ 100
- Intensity scan = 20mins*10 ≈ 3 hrs

Based on agreement between coincidence and covariance

- Question: why it seems to work so well?

Allum, Felix, et al. "Multi-Particle Three-Dimensional Covariance Imaging:“Coincidence” Insights into the Many-Body Fragmentation of Strong-Field Ionized D2O." *The Journal of Physical Chemistry Letters* 12 (2021): 8302-8308.

- Coincidence analysis
- Agreement between coincidence and covariance

why?

- Expectation of coincidence (math)
- Constraints in speed

2. Expectation in coincidence analysis

- The yield of (e, i_1) events should follow:

Normalized channel branching ratio

$$w_{coin} = \nu_0 * f(e, i_1) * \xi_e \, \xi_{i1}$$

Detection efficiency for electrons and ions

Rate of all events

2. Expectation in coincidence analysis

- The yield of (e, i_1) events should follow:

$$w^{(t)}(M, E) = f(M) f_M(E)\, p,$$
$$p = \xi_i \xi_e \nu_0\, \mathrm{e}^{-q},$$
$$q = \nu_0 (\xi_i + \xi_e - \xi_i \xi_e)$$

VS

Normalized channel branching ratio

$$w_{coin} = \nu_0 * f(e, i_1) * \xi_e\, \xi_{i1}$$

Detection efficiency for electrons and ions

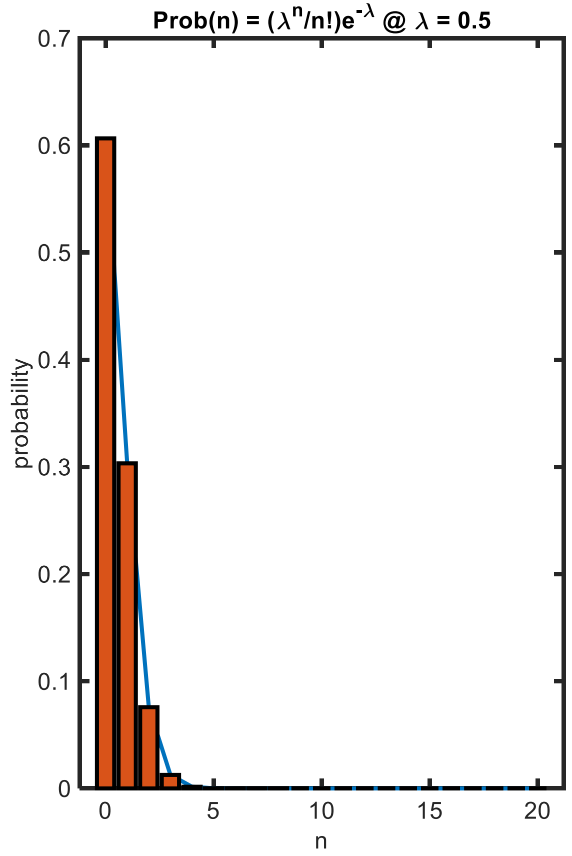
Rate of all events

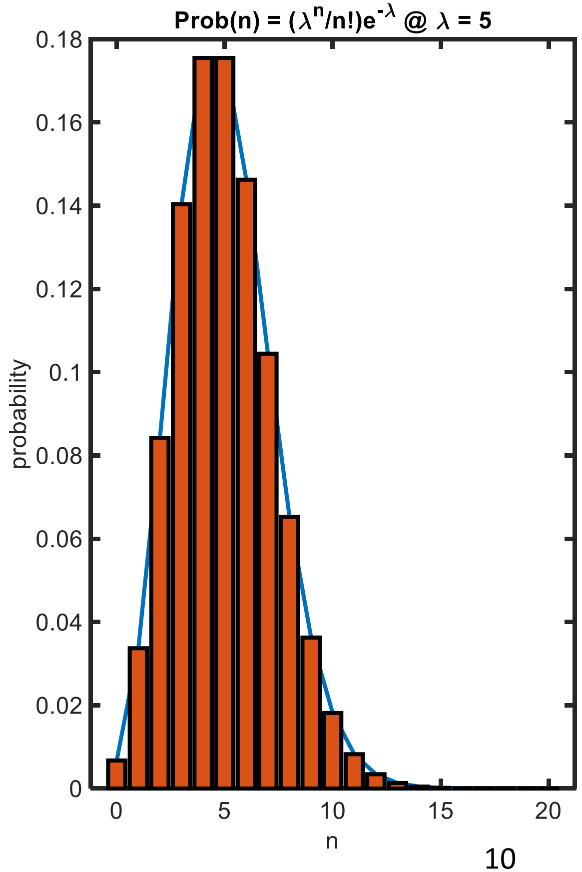
Mikosch, Jochen, and Serguei Patchkovskii. "Coincidence and covariance data acquisition in photoelectron and-ion spectroscopy. I. Formal theory." *Journal of Modern Optics* 60.17 (2013): 1426-1438.

2. Expectation in coincidence analysis

- In the exp, the data (ionization event) may fluctuate following Poisson distribution:

Shot #	e-	H+
1	1	1
2	0	0
3	1	1
4	1	1
5	0	0
6	3	3
7	1	1
8	0	0
…	…	…





2. Expectation in coincidence analysis

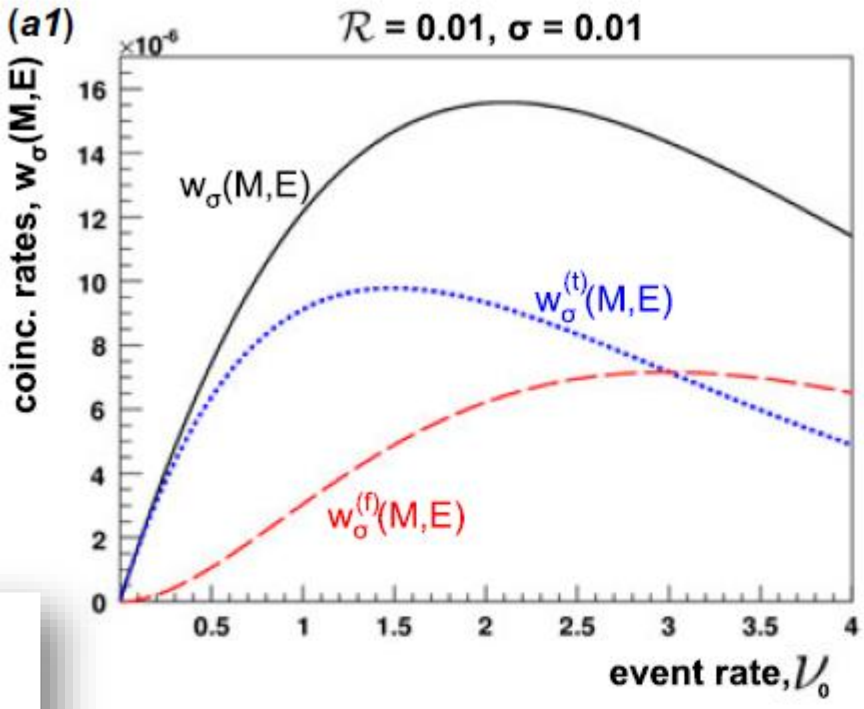
- In reality, there are more fragments (different events) in each shot

Shot #	e-	H+	O+	OH+	H2O+	…
1	3(1)	1	1	0	1	…
2	1(0)	0	0	0	1	…
3	1(1)	1	0	0	0	…
4	7(1)	1	1	0	5	…
5	3(0)	0	0	1	2	…
6	9(3)	3	0	2	4	…
7	7(1)	1	1	2	3	…
8	7(0)	0	0	0	7	…
…	…	…	…	…	…	…

2. Expectation in coincidence analysis

$$w(M,E) = true + false = w^{(t)}(M,E) + w^{(f)}(M,E)$$

- All = true + false
- Event rate -> intermediate ν_0



$$w^{(t)}(M,E) = f(M)\,f_M(E)\,p,$$
$$p = \xi_i \xi_e \nu_0\,\mathrm{e}^{-q},$$
$$q = \nu_0\,(\xi_i + \xi_e - \xi_i \xi_e)$$

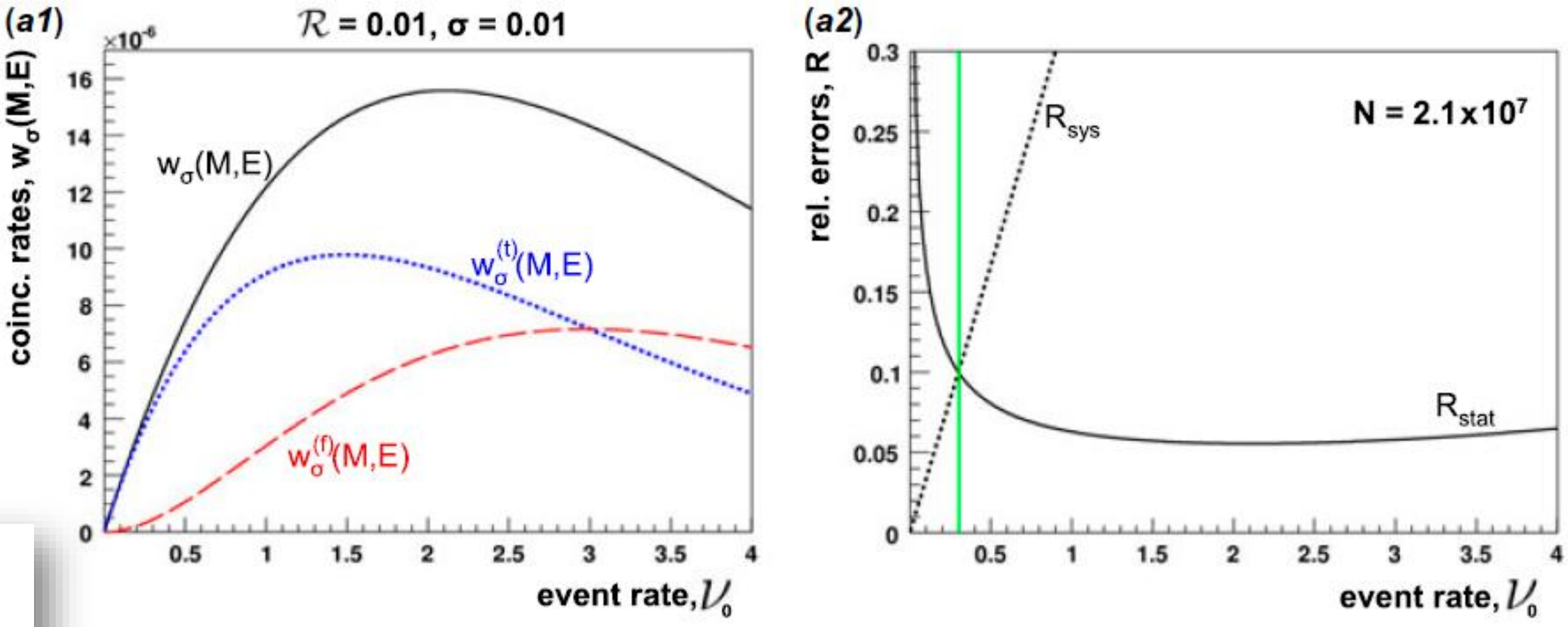
Mikosch, Jochen, and Serguei Patchkovskii. "Coincidence and covariance data acquisition in photoelectron and-ion spectroscopy. II. Analysis and applications." *Journal of Modern Optics* 60.17 (2013): 1439-1451.

2. Expectation in coincidence analysis

$$w(M,E) = true + false = w^{(t)}(M,E) + w^{(f)}(M,E)$$

$$R_{sys} = \frac{false}{all} = \frac{w^{(f)}(M,E)}{w(M,E)}$$

- All = true + false
- Event rate -> intermediate ν_0
- Sys error -> small ν_0



$$w^{(t)}(M,E) = f(M)\, f_M(E)\, p,$$
$$p = \xi_i \xi_e \nu_0\, \mathrm{e}^{-q},$$
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2. Expectation in coincidence analysis

- Typical acquisition times while keep low sys error:
- 1. Coincidence for (e, i) takes ~ 1hr
- 2. Coincidence for $(2e, 2i)$ takes ~ 10 days

- Solutions:
- 1. Go for higher repetition rate -> 100kHz (currently 1kHz) -> more shots per time
- 2. Go for clever analysis -> covariance analysis

Shot #	e-	H+	O+	OH+	H2O+	…
1	3(1)	1	1	0	1	…
2	1(0)	0	0	0	1	…
3	1(1)	1	0	0	0	…
4	7(1)	1	1	0	5	…
5	3(0)	0	0	1	2	…
6	9(3)	3	0	2	4	…
7	7(1)	1	1	2	3	…
8	7(0)	0	0	0	7	…
…	…	…	…	…	…	…

- Coincidence analysis
- Agreement between coincidence and covariance

why?

- Expectation of coincidence (math)
- Constraints in speed

- Expectation of covariance (math)
- Example of ToF-ToF

3. Carry out covariance – brief math Poisson distribution

- For Poisson distribution, the expectation values are:
- $P(N) = \frac{\lambda^N}{N!} e^{-\lambda}, P(N = 1) = \lambda e^{-\lambda}$
- $\langle N \rangle = \sum NP(N) = \sum N * \frac{\lambda^N}{N!} e^{-\lambda} = \lambda$
- $\langle N^2 \rangle = \sum N^2 P(N) = \sum N^2 * \frac{\lambda^N}{N!} e^{-\lambda} = \lambda^2 + \lambda$
- $Cov(N, N) = Var(N) = \langle N^2 \rangle - \langle N \rangle^2 = \lambda$

$$w^{(t)}(M, E) = f(M) f_M(E)\, p,$$
$$p = \xi_i \xi_e \nu_0\, \mathrm{e}^{-q},$$
$$q = \nu_0 (\xi_i + \xi_e - \xi_i \xi_e)$$

VS

w_{covar}

Mikosch, Jochen, and Serguei Patchkovskii. "Coincidence and covariance data acquisition in photoelectron and-ion spectroscopy. I. Formal theory." *Journal of Modern Optics* 60.17 (2013): 1426-1438.

3. Carry out covariance – brief math

$$\mathrm{Cov}(N_M, N_E) = \langle N_M N_E \rangle - \langle N_M \rangle \langle N_E \rangle$$
$$= P_1 \nu_0 + \sigma^2 \nu_0^2 (P_1 + P_2)(P_1 + P_3).$$

$$P_1 = f(M) f_M(E) \xi_i \xi_e$$

Term proportional to fluctuation.
For our laser it is small (~2%).

$$w^{(t)}(M, E) = f(M) f_M(E)\, p,$$
$$p = \xi_i \xi_e \nu_0\, \mathrm{e}^{-q},$$
$$q = \nu_0 (\xi_i + \xi_e - \xi_i \xi_e)$$

VS

w_{covar}

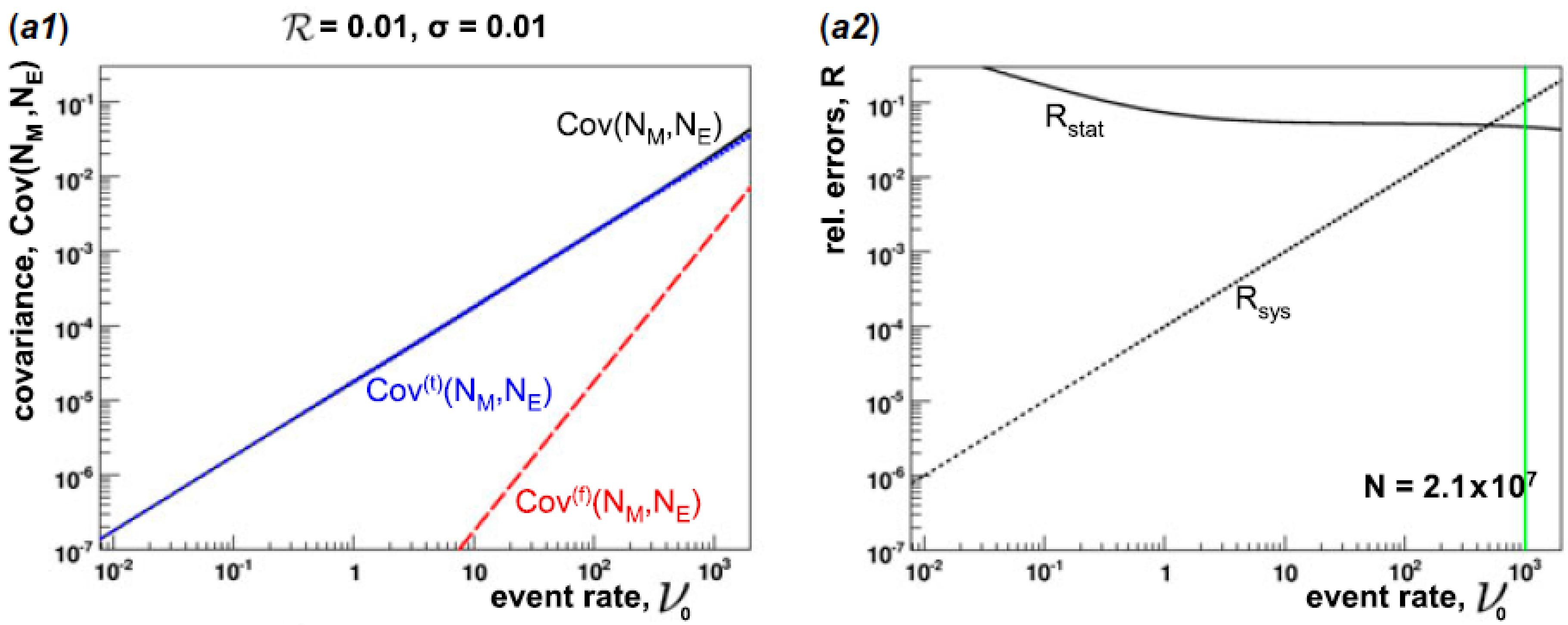
Mikosch, Jochen, and Serguei Patchkovskii. "Coincidence and covariance data acquisition in photoelectron and-ion spectroscopy. I. Formal theory." *Journal of Modern Optics* 60.17 (2013): 1426-1438.

3. Carry out covariance – simulation

$$w(M,E) = true + false = w^{(t)}(M,E) + w^{(f)}(M,E)$$

$$R_{sys} = \frac{false}{all} = \frac{w^{(f)}(M,E)}{w(M,E)}$$

- All = true + false
- Event rate -> the higher ν_0 the better!!
- Sys error -> large ν_0!!



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3. Carry out covariance – e.g. ToFToF

$$Cov(N^a, N^b) \overset{\text{def}}{=} \langle N^{t_1} N^{t_2} \rangle - \langle N^{t_1} \rangle \langle N^{t_2} \rangle$$

i denotes shot number

$$= \frac{\Sigma N_i^{t_1} N_i^{t_2}}{Shots} - \frac{\Sigma N_i^{t_1} \Sigma N_i^{t_2} - \Sigma N_i^{t_1} N_i^{t_2}}{Shots(Shots - 1)}$$

$$= \frac{\Sigma N_i^{t_1} N_i^{t_2}}{Shots - 1} - \frac{\Sigma N_i^{t_1} \Sigma N_i^{t_2}}{Shots(Shots - 1)}$$

$$\approx \frac{\Sigma N_i^{t_1} N_i^{t_2}}{Shots} - \frac{\Sigma N_i^{t_1} \Sigma N_i^{t_2}}{Shots^2}$$

Goal is to compute these two terms summed over i

3. Carry out covariance – e.g. ToFToF

Language of computing the fluctuation

$$= \langle (N^{t_1} - \langle N^{t_1} \rangle)\,(N^{t_2} - \langle N^{t_2} \rangle) \rangle$$

$$Cov(N^a, N^b) = \langle\, N^{t_1} N^{t_2} \rangle - \langle\, N^{t_1} \rangle \langle N^{t_2} \rangle$$

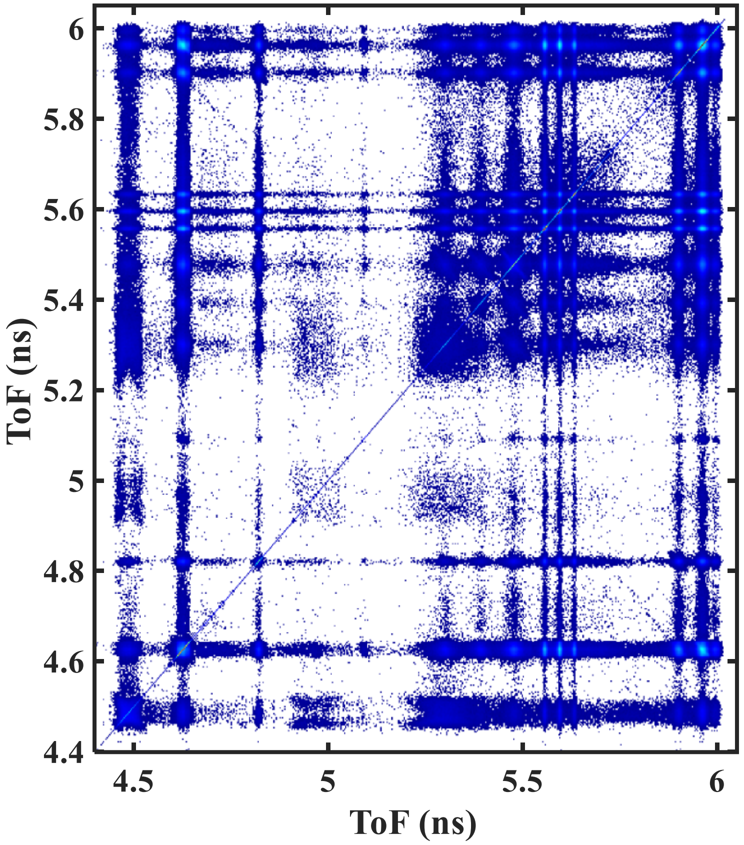
$$= \frac{\Sigma N_i^{t_1} N_i^{t_2}}{Shots} - \frac{\Sigma N_i^{t_1} \Sigma N_i^{t_2} - \Sigma N_i^{t_1} N_i^{t_2}}{Shots(Shots - 1)}$$

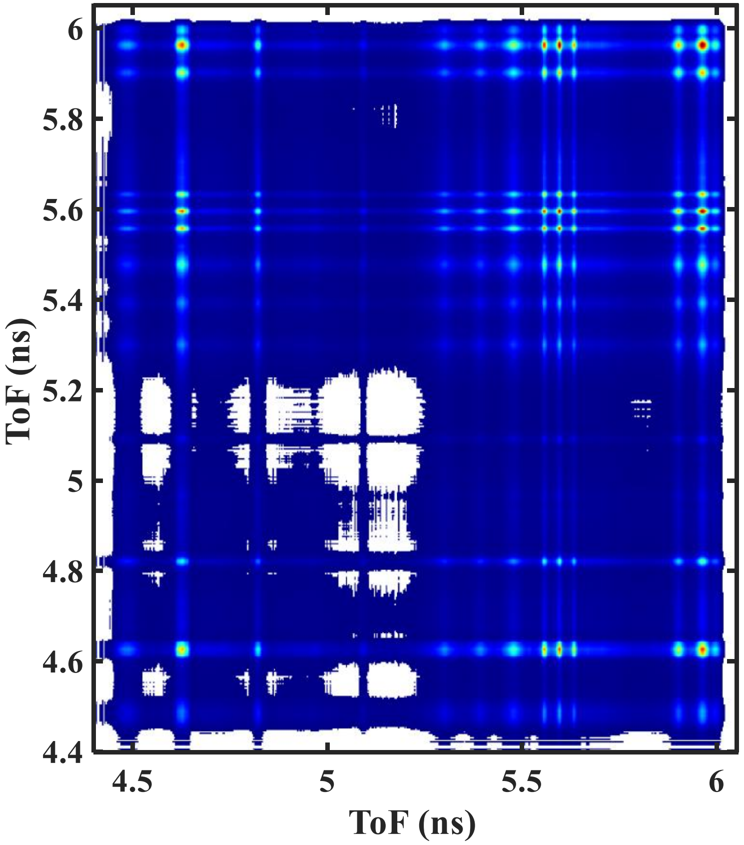
$$= \frac{\Sigma N_i^{t_1} N_i^{t_2}}{Shots - 1} - \frac{\Sigma N_i^{t_1} \Sigma N_i^{t_2}}{Shots(Shots - 1)}$$

$$\approx \frac{\Sigma N_i^{t_1} N_i^{t_2}}{Shots} - \frac{\Sigma N_i^{t_1} \Sigma N_i^{t_2}}{Shots^2}$$

3. Carry out covariance – e.g. ToFToF

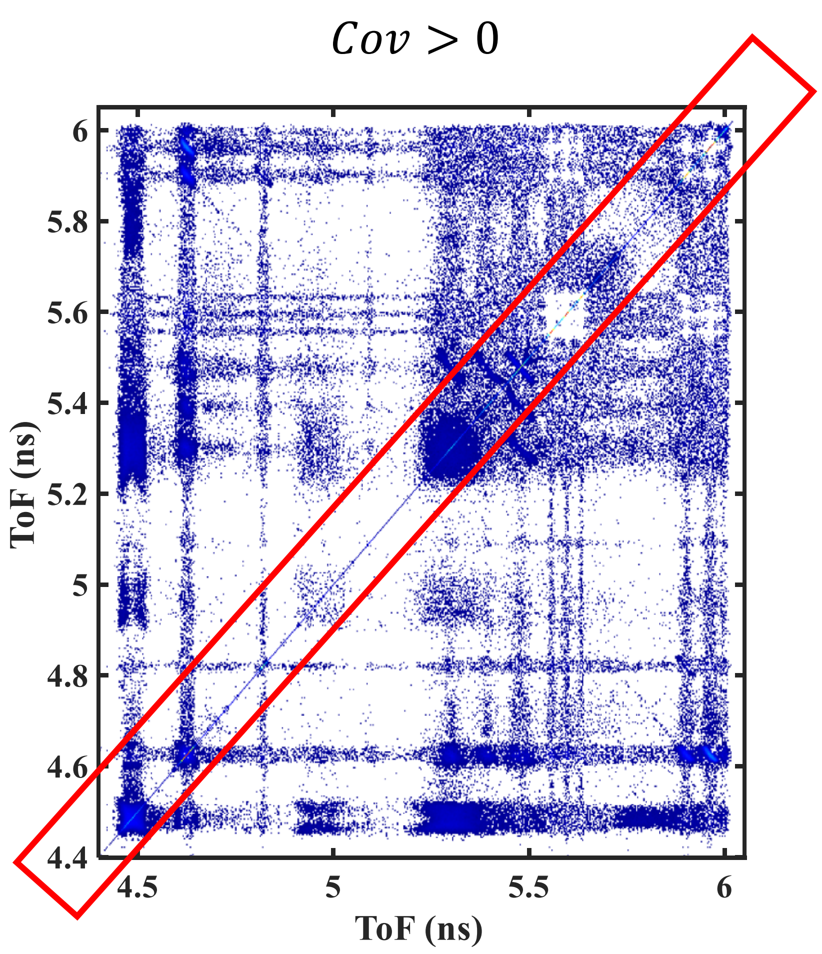
$$S_{12} = \frac{\Sigma N_i^{t_1} N_i^{t_2}}{Shots} \quad - \quad S_1 S_2 = \frac{\Sigma N_i^{t_1} \Sigma N_i^{t_2}}{Shots^2}$$

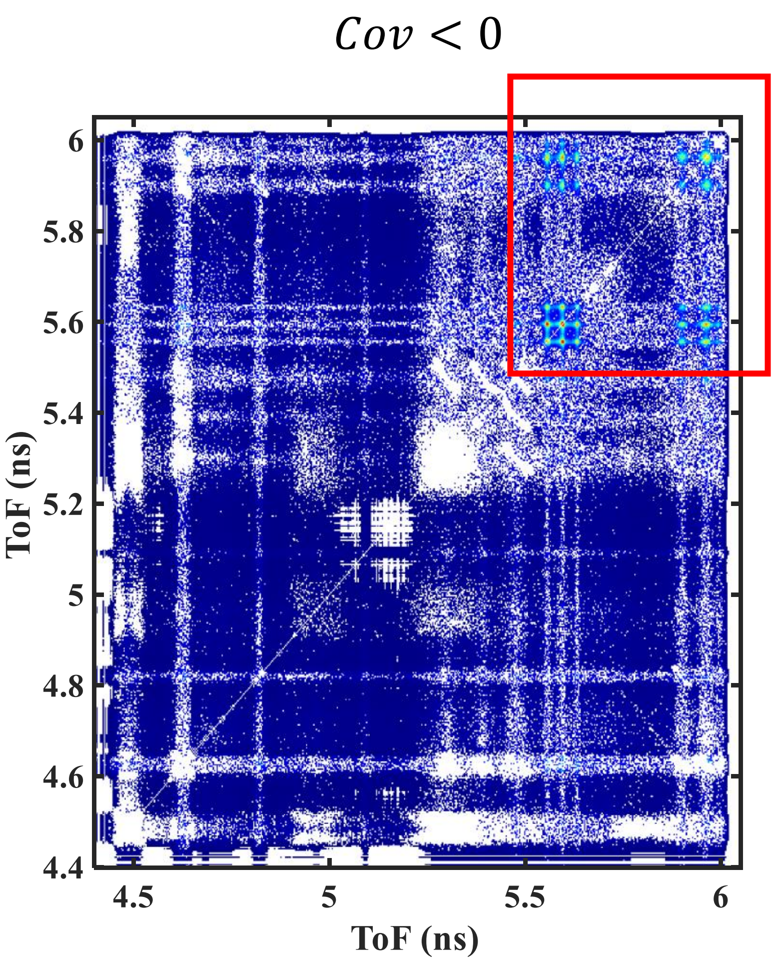




3. Carry out covariance – e.g. ToFToF

- Diagonal
- Negative





3. Carry out covariance – diagonal: t1 = t2 = t

$$
\begin{aligned}
Cov(N^{(a)}, N^{(a)}) &= \langle N^{t_1} N^{t_2} \rangle - \langle N^{t_1} \rangle \langle N^{t_2} \rangle \\
&= \frac{\Sigma (N_i^t)^2}{Shots} - \frac{\Sigma N_i^t \Sigma N_i^t - \Sigma (N_i^t)^2}{Shots(Shots-1)} \\
&= \frac{\Sigma (N_i^t)^2}{Shots-1} - \frac{\Sigma N_i^t \Sigma N_i^t}{Shots(Shots-1)}
\end{aligned}
$$

Also counting identical pairs

An event rate is λ, produces k fragments that has identical detection efficiency η:

$$
\begin{aligned}
\langle N \rangle &= \lambda k \eta \\
Cov(N, N) &= \lambda (2 C_k^2 \eta^2 + k\eta)
\end{aligned}
$$

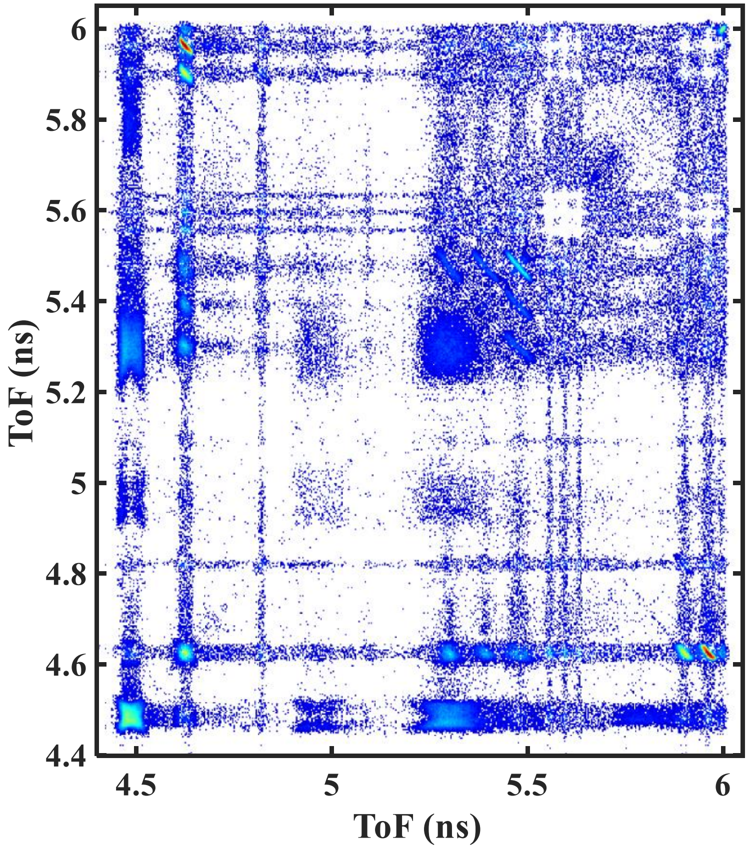
Contribution of recounting

$$
\begin{aligned}
&Cov(N^{(a)}, N^{(a)}) \rightarrow Cov(N^{(a)}, N^{(a)}) - \langle N^{(a)} \rangle \\
&Cov(N^{(a)}, N^{(b)}) \rightarrow Cov(N^{(a)}, N^{(b)}) - \langle N^{(a)} \cap N^{(b)} \rangle
\end{aligned}
$$

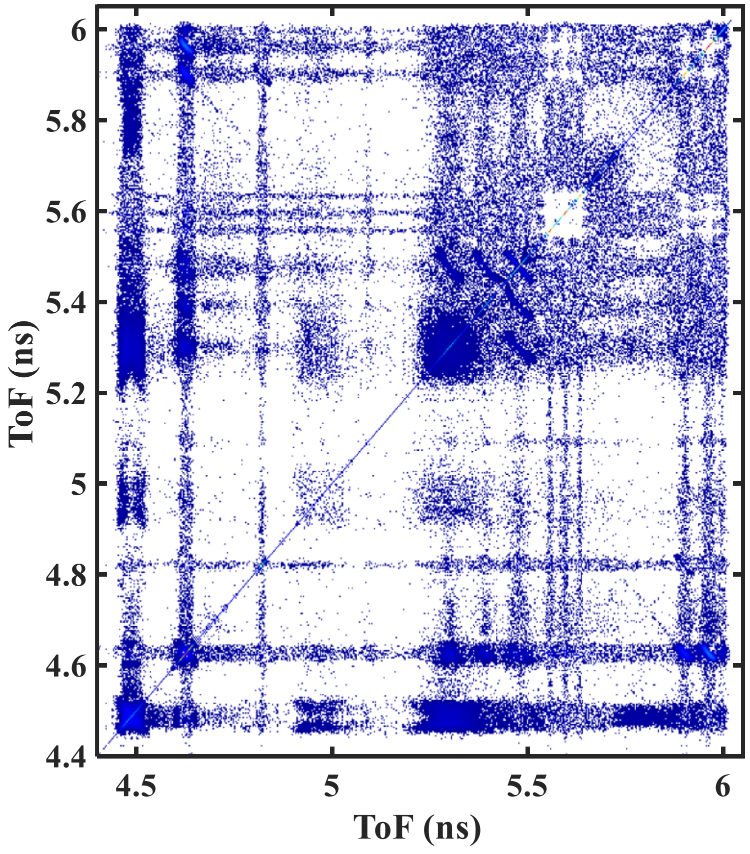
3. Carry out covariance – e.g. ToFToF

- Diagonal
- Negative

$$Cov(N^a, N^b) - \langle N^a \cap N^b \rangle > 0$$

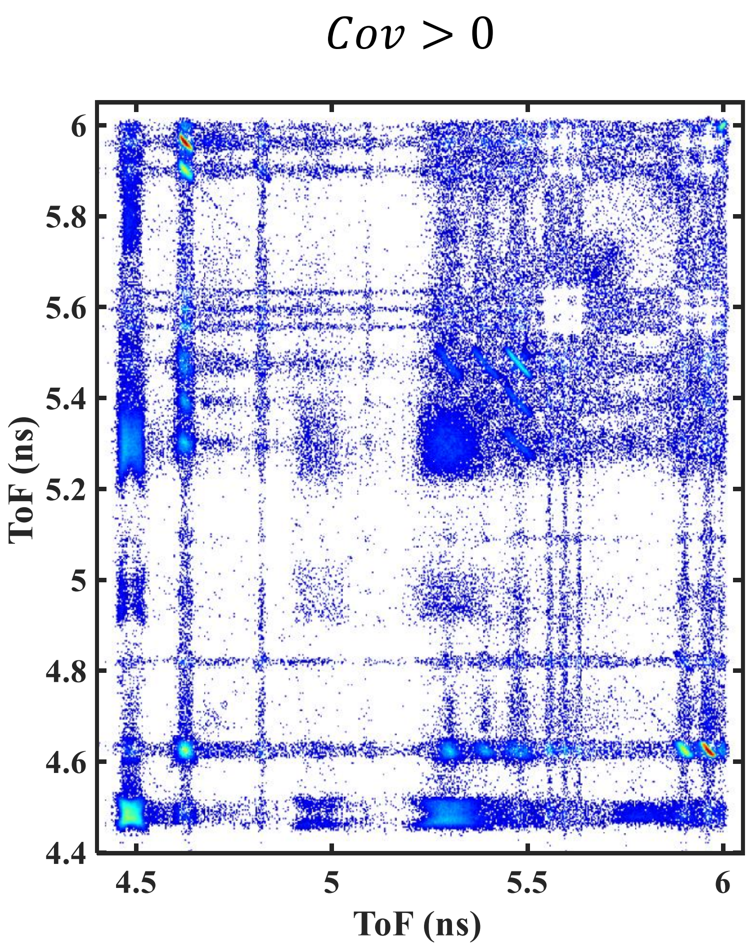


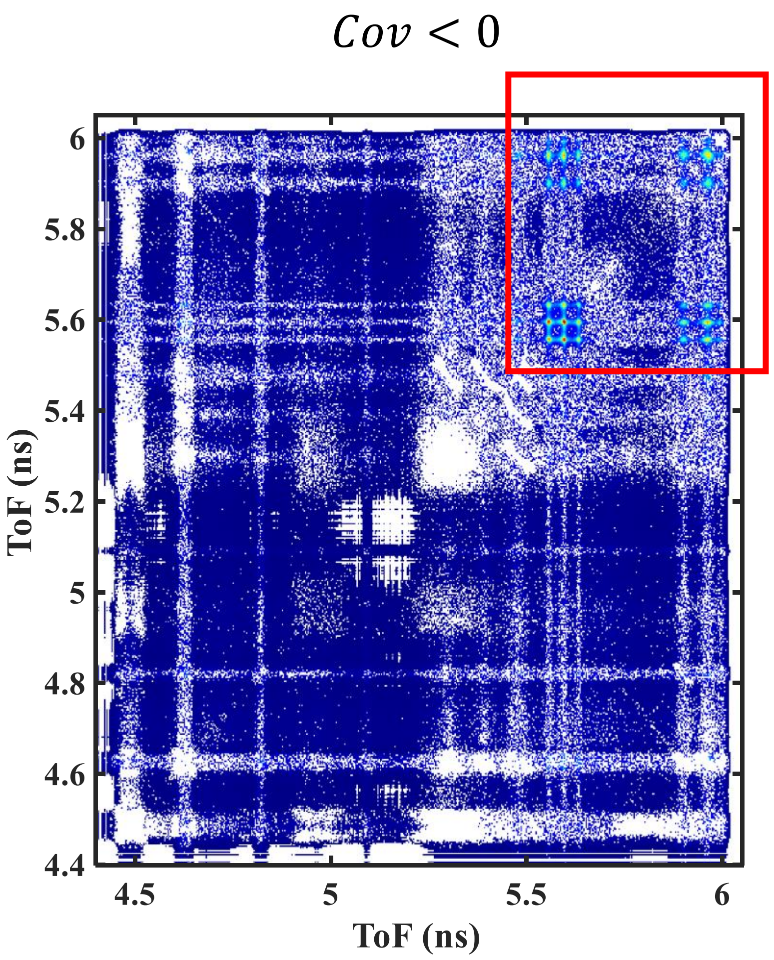
$$Cov(N^a, N^b) > 0$$

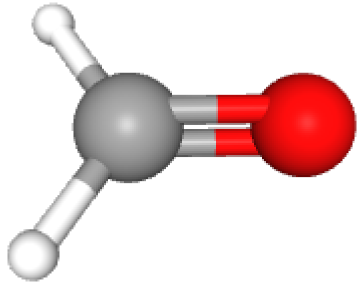


3. Carry out covariance – e.g. ToFToF

- Diagonal
- Negative



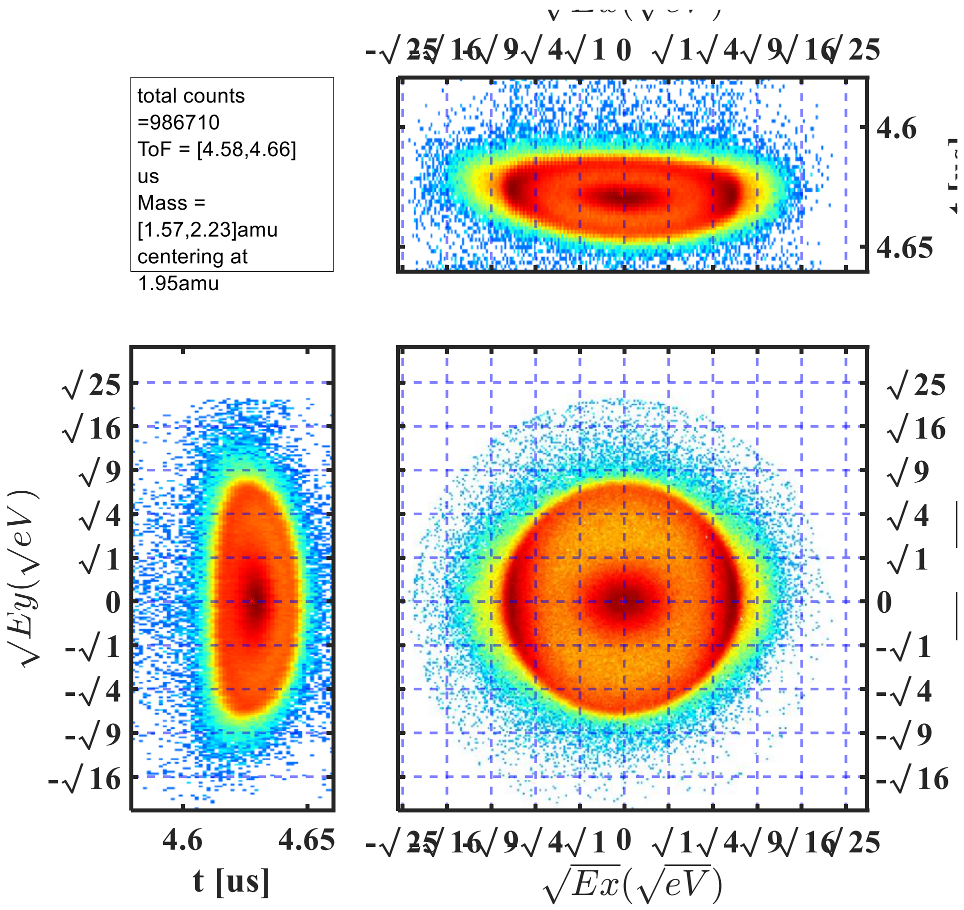




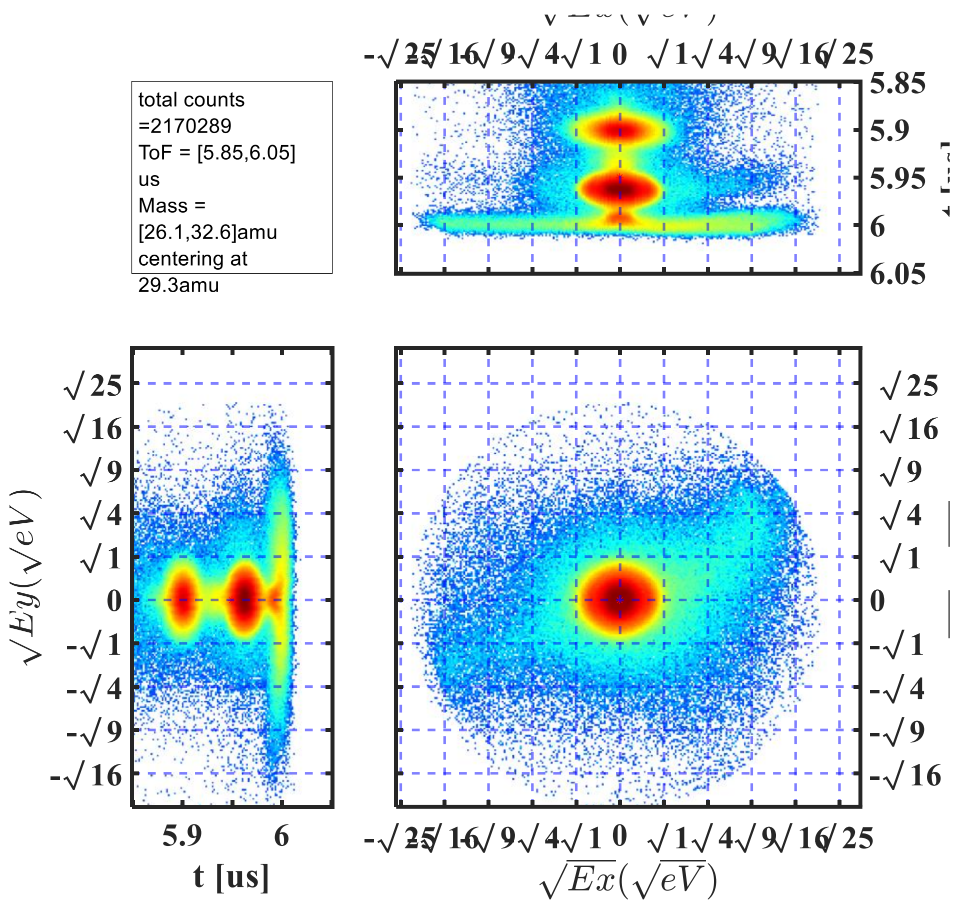
CD_2O

3. Carry out covariance – negative parts of covariance

D^+: x-y, x-t, y-t plots



CO^+, CDO^+, CD_2O^+: x-y, x-t, y-t plots



- Single ionization dissociation low KE
- $p_1 + p_2 = 0$
- $KE_1 : KE_2 = m_2 : m_1$
- Overlapping hits issue

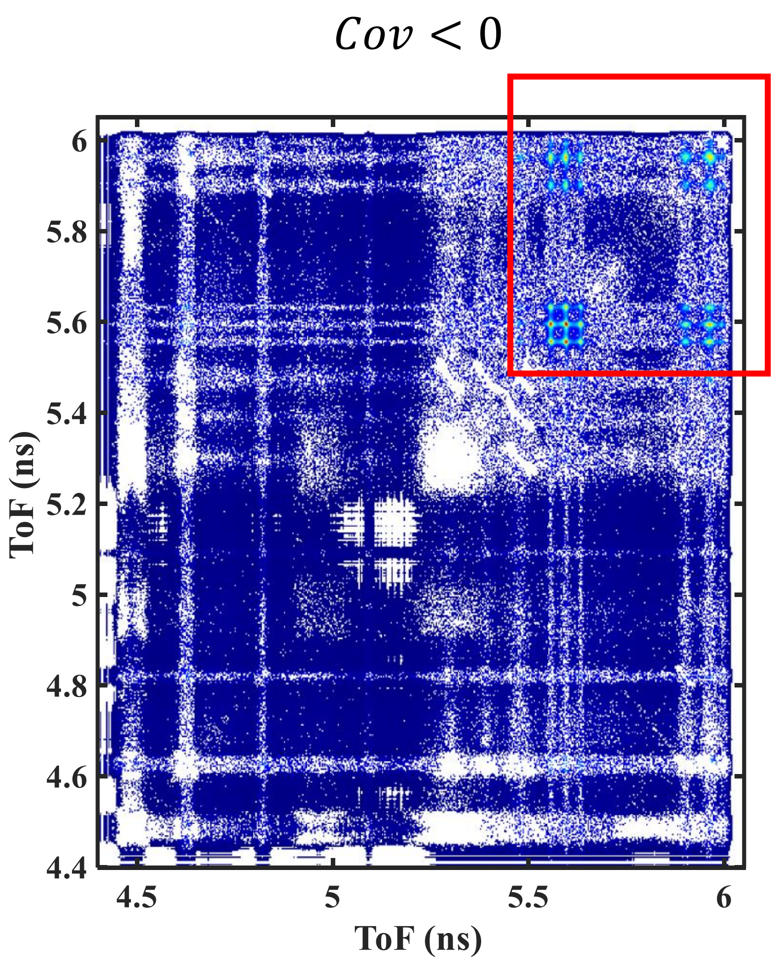
3. Carry out covariance – negative parts of covariance

$$p(N) = \begin{cases} 1 - p = p_1(0) & N = 0 \\ p = \Sigma_{n \geq 1} p_1(n) & N = 1 \end{cases}$$

And the related statistics are:

$$\langle N \rangle = p$$
$$Cov(N, N) = p(1 - p)$$
$$Cov(N, N) - \langle N \rangle = -p^2$$

- Any imaging-centroiding detector will have this problem



- Coincidence analysis
- Agreement between coincidence and covariance

why?

- Expectation of coincidence (math)
- Constraints in speed

- Expectation of covariance (math)
- Example of ToF-ToF

- Other covariance, example of our data (CD_2O)
- Related thoughts

4. More about covariance – matrix

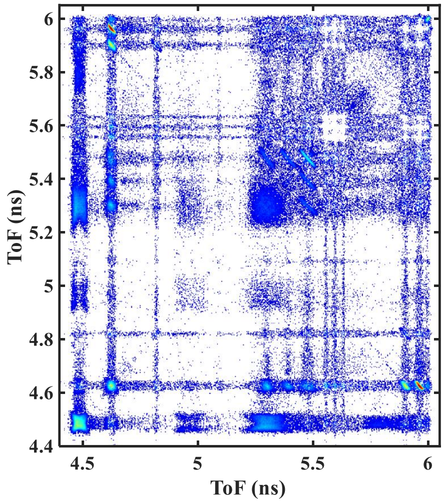
- $Cov(N^{i_1}, N^{i_2})$
- Matrix of covariance
- $Cov\left(N^{i_1:t_1}, N^{i_2:t_2}\right)$
- $N^{i_1} = \sum_{t_1} N^{i_1:t_1}$
- Other observables like x, y

From Wikipedia:

$$\mathrm{K}_{\mathbf{XX}} = \begin{bmatrix} \mathrm{E}[(X_1 - \mathrm{E}[X_1])(X_1 - \mathrm{E}[X_1])] & \mathrm{E}[(X_1 - \mathrm{E}[X_1])(X_2 - \mathrm{E}[X_2])] & \cdots & \mathrm{E}[(X_1 - \mathrm{E}[X_1])(X_n - \mathrm{E}[X_n])] \\ \mathrm{E}[(X_2 - \mathrm{E}[X_2])(X_1 - \mathrm{E}[X_1])] & \mathrm{E}[(X_2 - \mathrm{E}[X_2])(X_2 - \mathrm{E}[X_2])] & \cdots & \mathrm{E}[(X_2 - \mathrm{E}[X_2])(X_n - \mathrm{E}[X_n])] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}[(X_n - \mathrm{E}[X_n])(X_1 - \mathrm{E}[X_1])] & \mathrm{E}[(X_n - \mathrm{E}[X_n])(X_2 - \mathrm{E}[X_2])] & \cdots & \mathrm{E}[(X_n - \mathrm{E}[X_n])(X_n - \mathrm{E}[X_n])] \end{bmatrix}$$

The definition above is equivalent to the matrix equality

$$\mathrm{K}_{\mathbf{XX}} = \operatorname{cov}[\mathbf{X}, \mathbf{X}] = \mathrm{E}[(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{X} - \mu_{\mathbf{X}})^{\mathrm{T}}] = \mathrm{E}[\mathbf{X}\mathbf{X}^T] - \mu_{\mathbf{X}}\mu_{\mathbf{X}}^T \quad \text{(Eq.1)}$$

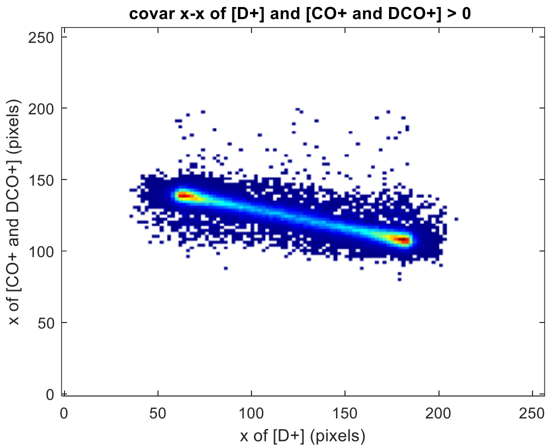


4. More about covariance – momentum correlation

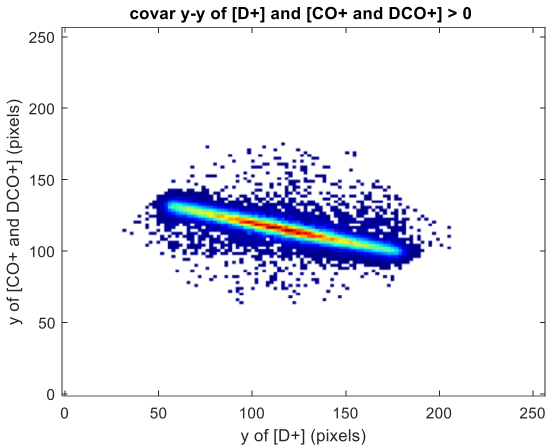
Particle 1 choose D^+, look at its x direction
Particle 2 choose CO^+/DCO^+, look at its x direction

- $Cov\left(N^{D^+:x}, N^{CO^+/DCO^+:x}\right)$
- Momentum conservation
- Coincidence feature
- Angular distribution

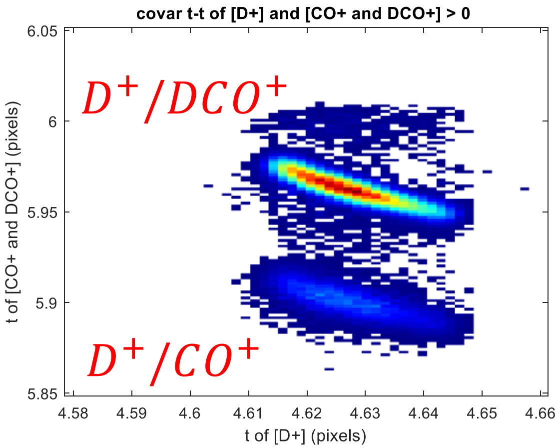
x-x



y-y



t-t

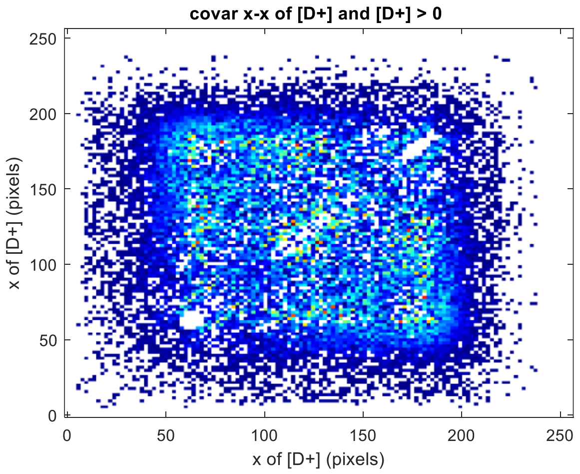


4. More about covariance – momentum correlation

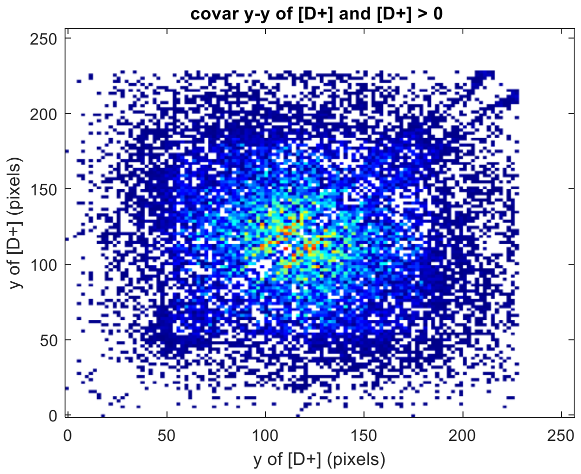
Particle 1 choose D^{+}, look at its x direction
Particle 2 choose D^{+}, look at its x direction

- $Cov(N^{D^{+}:x}, N^{D^{+}:x})$
- 3-body or 4-body dissociation
- Diagonal missing

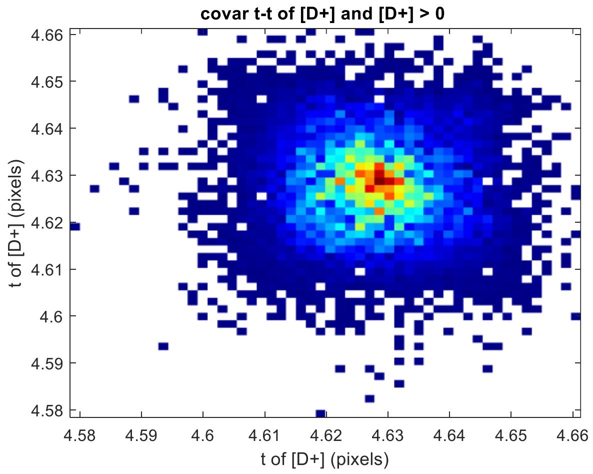
x-x



y-y



t-t



4. More about covariance – $Cov(N_{xy}, N)$

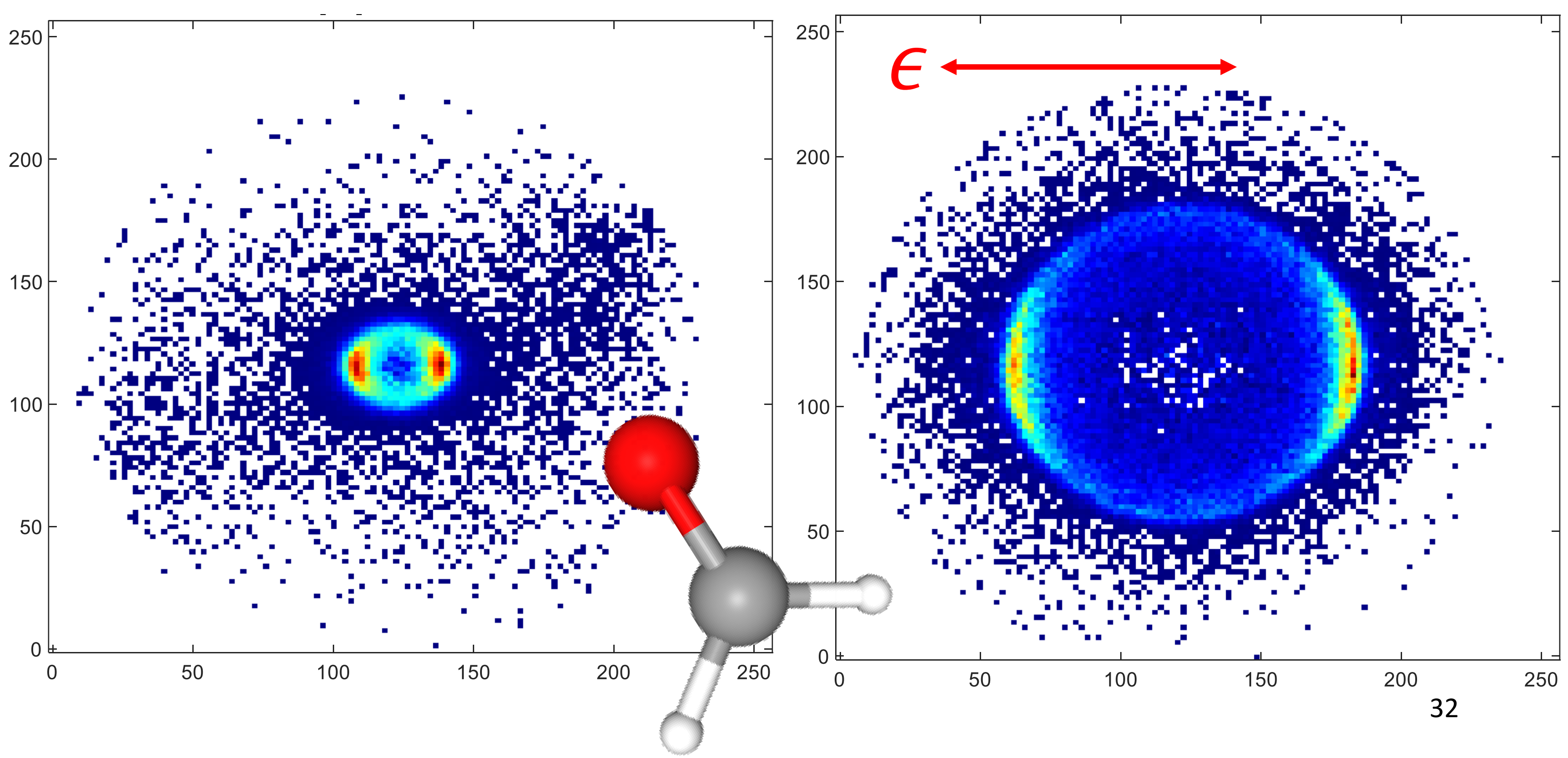
Particle 1 choose CO^+/DCO^+, look at its xy image
Particle 2 choose D^+, look at its $sum\ yield$

Particle 1 choose CO^+/DCO^+, look at its $sum\ yield$
Particle 2 choose D^+, look at its xy image

$Cov(N^{CO^+/DCO^+:xy}, N^{D^+})$

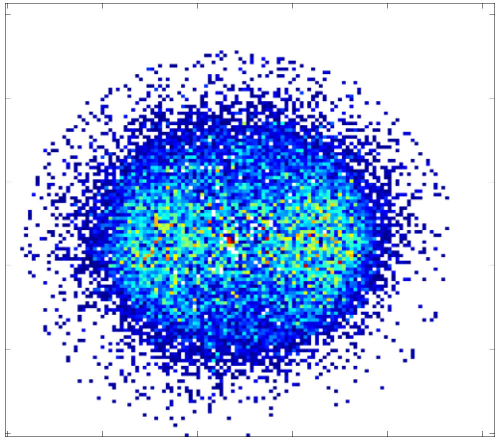
$Cov(N^{CO^+/DCO^+}, N^{D^+:xy})$

- $p_1 + p_2 = 0$
- $KE_1 : KE_2 = m_2 : m_1$
- 2-body dissociation
- Angular distribution (alignment)

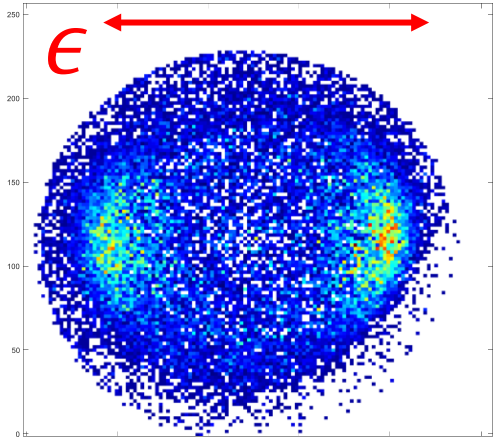


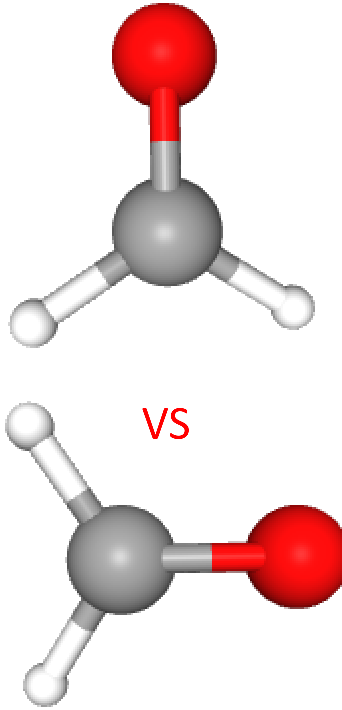
4. More about covariance – 3-body dissociation

$Cov(N^{O^+:xy}, N^{D^+})$



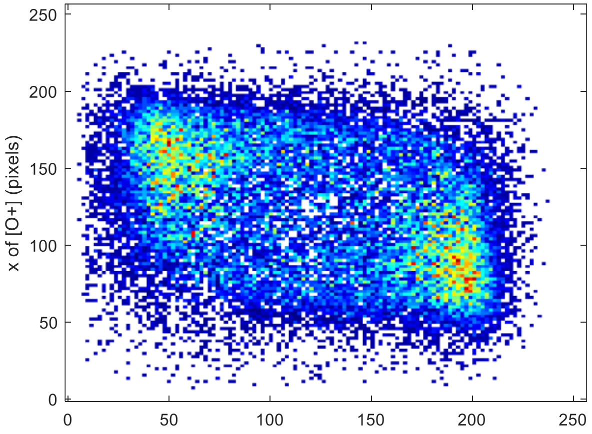
$Cov(N^{O^+}, N^{D^+:xy})$



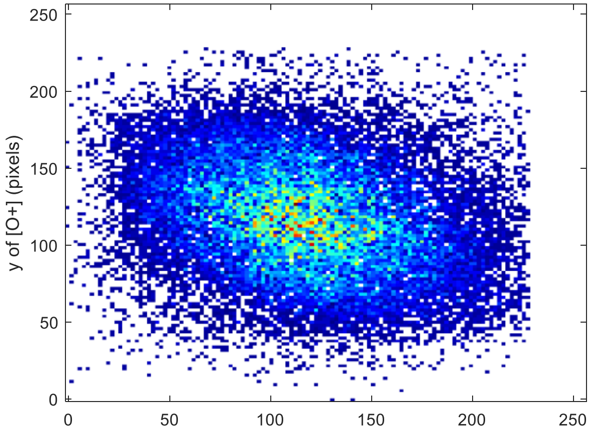


- vague angular distribution
- No clear momentum conservation
- Has to do 3-body covariance

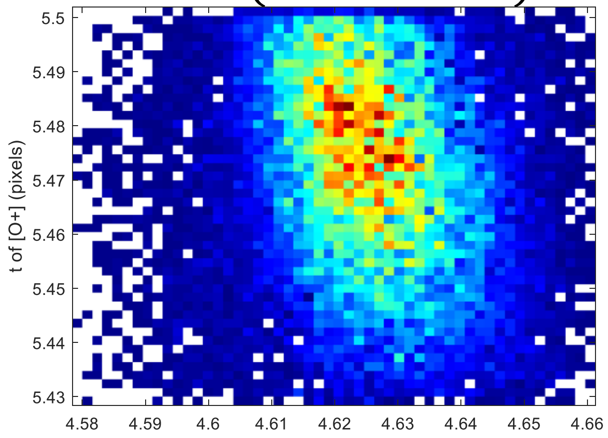
$Cov(N^{O^+:x}, N^{D^+:x})$



$Cov(N^{O^+:y}, N^{D^+:y})$



$Cov(N^{O^+:t}, N^{D^+:t})$



4. More about covariance – compared to coincidence

fast

Coincidence: (x, y, t) list → preprocess → $(x_1, y_1, t_1; x_2, y_2, t_2)$ list → Further analysis, usually histograms → { Momentum correlation, Angular distribution, … }

lossy

Depending on details, ~ 10s to 10mins

Covariance: (x, y, t) list → Specific covariance code → { Momentum correlation, Angular distribution, … }

We should be doing covariance 10 years ago! O(∩_∩)O

4. More about covariance – related ideas

- The pressure should be proportional to the event rate so $Cov(N_e(t), P\ ) \propto \langle N_e \rangle$?

- For a pump probe system like UV-VUV, would it be possible if: $Cov(N_e(t), I_{UV} I_{VUV}) \propto \langle N_e \rangle$ assuming $N_e(t) \propto I_{UV} I_{VUV}$?

- Use ion counts together with photoelectrons

- Partial covariance may help clean some noise as well

- Thanks for joining. Any thoughts or questions?

Check out our paper:

Allum, Felix, et al. "Multi-Particle Three-Dimensional Covariance Imaging:“Coincidence” Insights into the Many-Body Fragmentation of Strong-Field Ionized D2O." *The Journal of Physical Chemistry Letters* 12 (2021): 8302-8308.

- Coincidence analysis
- Agreement between coincidence and covariance

why?

- Expectation of coincidence (math)
- Constraints in speed

- Expectation of covariance (math)
- Example of ToF-ToF

- Other covariance, example of our data (CD_2O)
- Related thoughts

Check out our paper:

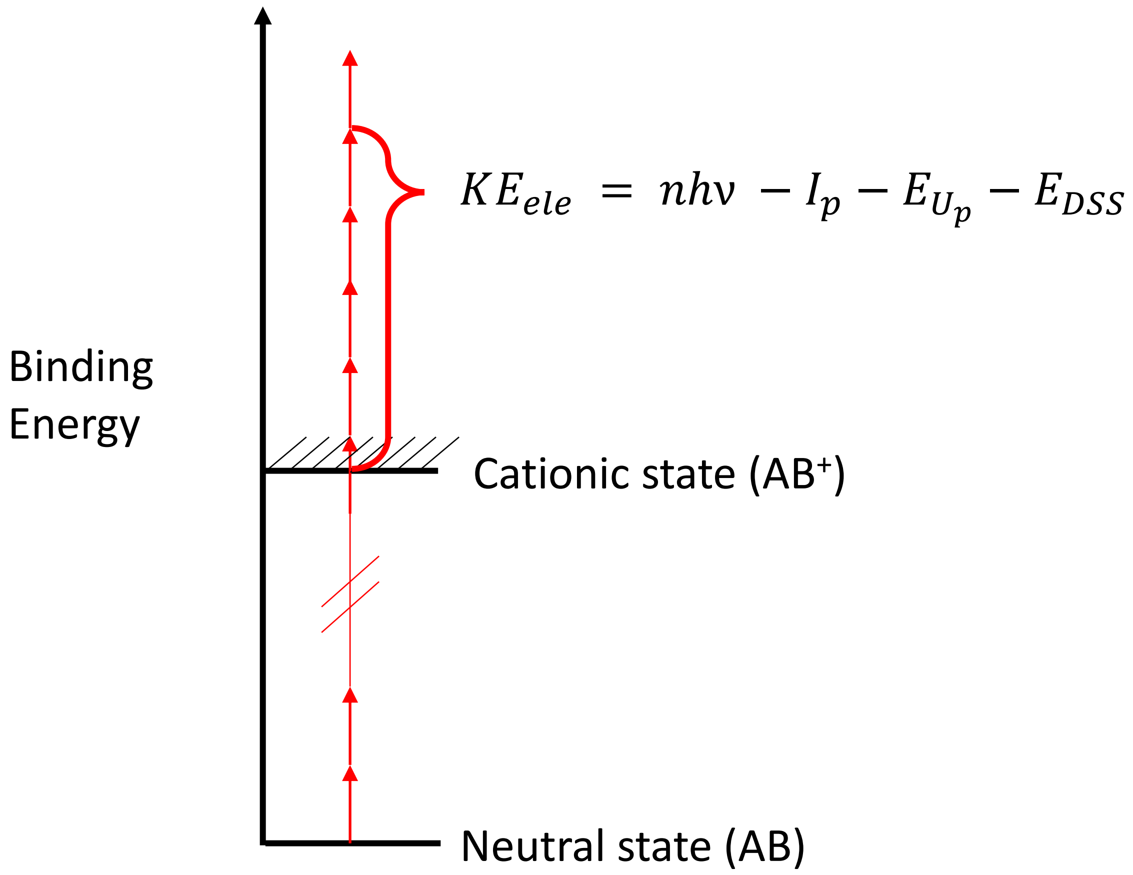
Allum, Felix, et al. "Multi-Particle Three-Dimensional Covariance Imaging:“Coincidence” Insights into the Many-Body Fragmentation of Strong-Field Ionized D2O." *The Journal of Physical Chemistry Letters* 12 (2021): 8302-8308.

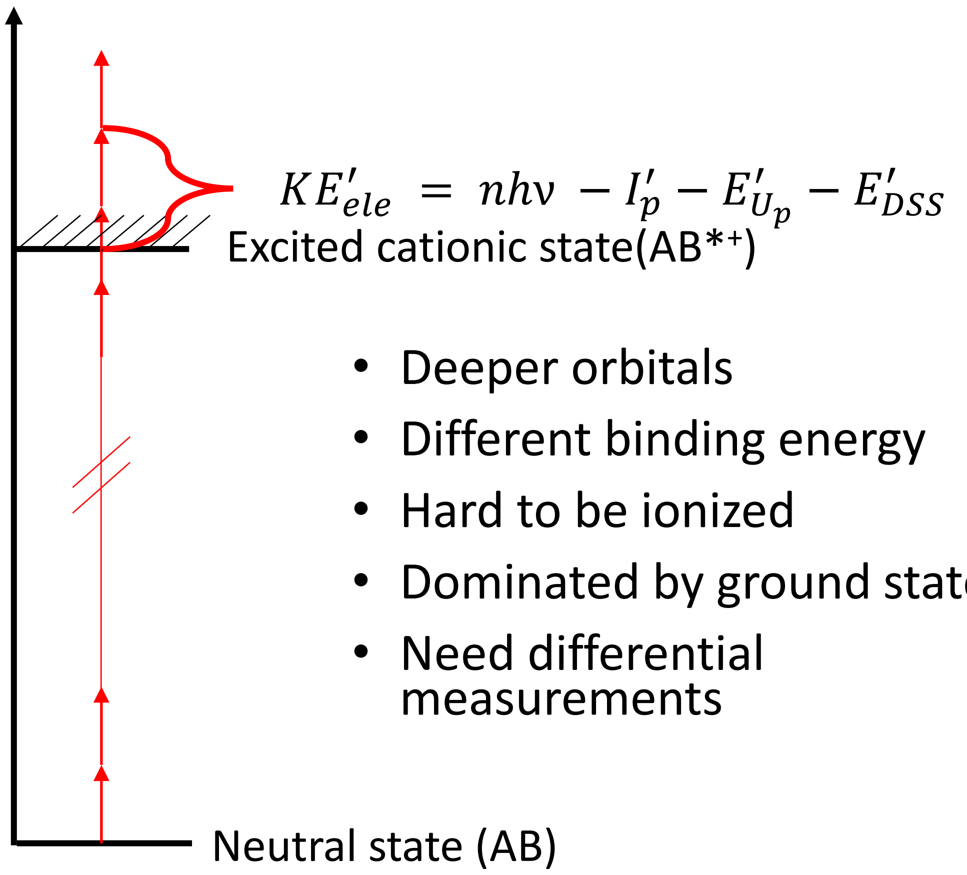
Some reading materials

https://journals.aps.org/pra/abstract/10.1103/PhysRevA.89.011401
https://journals.aps.org/pra/abstract/10.1103/PhysRevA.91.053424
https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.113.073005
https://aip.scitation.org/doi/full/10.1063/1.4947551
https://aip.scitation.org/doi/full/10.1063/1.5041381
https://pubs.rsc.org/en/content/articlehtml/2020/fd/d0fd00115e
https://www.nature.com/articles/s42004-020-0320-3
https://journals.aps.org/pra/abstract/10.1103/PhysRevA.89.053418

Mainly focus on the following papers:
https://www.tandfonline.com/doi/full/10.1080/09500340.2013.839837
https://www.tandfonline.com/doi/full/10.1080/09500340.2013.839840

Appendix 1: CRATI





- Deeper orbitals
- Different binding energy
- Hard to be ionized
- Dominated by ground state
- Need differential measurements

Appendix 3: ToFToF computation

$$\frac{\Sigma N_i^{t_1} N_i^{t_2}}{Shots}$$

```
for ii = 1:1:length(trigindex) - 1
    for jj1 = trigindex(ii):trigindex(ii + 1) - 1
         for jj2 = trigindex(ii):trigindex(ii + 1) - 1
            ToFToF(ToFindexes(jj1),ToFindexes(jj2)) = ToFToF(ToFindexes(jj1),ToFindexes(jj2)) + 1;
        end
    end
end
```

Appendix 4: more poisson

- Poisson distribution and multiple events
- Event n follows Poisson distribution:
- $P(N_n) = \frac{\lambda_n^{N_n}}{N_n!} e^{-\lambda_n}$
- Simple calculation shows that, summing up all m events to get everything $N = N_1 + N_2 + \ ... + N_m$, N follows:
- $P(N) = \frac{\lambda^N}{N!} e^{-\lambda}\ where\ \lambda = \lambda_1 + \lambda_2 + \ ... + \lambda_m$
- Which tells us that: everything is Poisson style.

Appendix 4: more poisson

- For Poisson distribution, the expectation value is:
- $< N > = \sum NP(N) = \sum N * \frac{\lambda^N}{N!} e^{-\lambda} = \lambda$
- $< N^2 > = \sum N^2 P(N) = \sum N^2 * \frac{\lambda^N}{N!} e^{-\lambda} = \lambda^2 + \lambda$
- $< N^3 > = \sum N^3 P(N) = \sum N^3 * \frac{\lambda^N}{N!} e^{-\lambda} = \lambda^3 + 3\lambda^2 + \lambda$
- $\sqrt{Var(N)} = \sqrt{< (N - < N >)^2 >} = \sqrt{< N^2 > - < N >^2} = \sqrt{\lambda}$
- $\frac{<N>}{\sqrt{Var(N)}} = \sqrt{\lambda}$

Appendix 4: more poisson

- Another view of the covariance: fluctuation between two particles
- Imagine our detection efficiency is 100% for both ele and ion.
- $Cov(N_e, N_M) = Cov(N_{M1} + N_{M2} + \ ..., N_M) = Cov(N_M, N_M) = N_M$

Expand the sources of electrons: with different ions

Nature of irrelevant events the covariance calculation. Note 1 ionization is not going to generate so many different ions!

Nature of Poisson style distribution

Appendix 4: more poisson

- Another view of the covariance: fluctuation between two particles
- Imagine our detection efficiency is 100% for both ele and ion.
- $Cov(N_e, N_M) = Cov(N_{M1} + N_{M2} + \ \ldots, N_M) = Cov(N_M, N_M) = N_M$

Note that with the help of N_M the N_e has been filtered out to only include events involve N_M!!

- $Cov(\xi_e N_e, \xi_i N_M) = \cdots = Cov(\xi_e N_M, \xi_i N_M) = \xi_e \xi_i N_M$

1. Strong of coincidence

- ~~0. the multiple particle nature of the measurement~~
- ~~1. MRATI~~
- ~~2. short VS long~~
- ~~3. example of covariance to fit coincidence~~

2. Coincidence expectation

- ~~1. from intuition~~
- ~~2. from real math~~
- ~~3. the boundary -> acquisition time (stat table) -> MRI and covariance~~

3. Carry out covariance

- ~~1. covar math brief~~
- ~~2. covar expectation plot~~
- ~~3. example: ToFToF~~

4. More about covariance

- ~~1. different covar (ToFToF, Nnxy , NxNy, betaKER) speed and back to coincidence~~
- ~~2. possible covar params/apps – UV with ions or laser intensity~~
- 3. covar constraints** -- comparison and speed boost math